Exercises Week 12: Optimization on nonsmooth sets through lifts

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1 Minimization on an interval

Let $f : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable. Consider the *constrained* minimization problem

$$\min_{x \in \mathbb{R}} f(x) \quad \text{subject to} \quad -1 \leq x \leq 1. \tag{P}$$

Alternatively, consider the *unconstrained* minimization problem

$$\min_{y \in \mathbb{R}} g(y) \quad \text{where} \quad g = f \circ \sin. \tag{Q}$$

1. Argue that (P) has at least one global minimizer.
2. Argue that the optimal values of (P) and (Q) are equal.

A second-order critical (SOC) point for the unconstrained minimization problem (Q) is a point $y \in \mathbb{R}$ such that $g'(y) = 0$ and $g''(y) \geq 0$.

3. Argue carefully that if y is second-order critical for (Q) then $x = \sin(y)$ is stationary for (P).

4. Now assume f is convex. Deduce that if y is a local minimizer of g then in fact y is a global minimizer of g.

2 Sphere-to-simplex lift

Let $\mathbb{S}^{d-1} = \{y \in \mathbb{R}^d \mid \|y\| = 1\}$ be the unit sphere in $\mathbb{R}^d$ and define the simplex

$$\Delta^{d-1} = \{y \in \mathbb{R}^d \mid y_i \geq 0 \text{ for all } i \text{ and } \sum_{i=1}^{d} y_i = 1\}.$$

The standard simplex plays a crucial role in various optimization problems encountered in fields such as data science, machine learning, statistics, and beyond.

Consider the sphere-to-simplex (or Hadamard) lift, given by

$$\varphi \colon \mathbb{S}^{d-1} \to \Delta^{d-1}, \quad \varphi(y) = y^{\odot 2} = (y_1^2, \ldots, y_d^2) \tag{sphere-to-simplex}$$

We say that the lift φ satisfies the $2 \Rightarrow 1$ property if for any smooth function $f \colon \mathcal{E} \to \mathbb{R}$ and $g = f \circ \varphi$, if $y \in \mathbb{S}^{d-1}$ is a second order critical point for the problem $\min_{y' \in \mathbb{S}^{d-1}} g(y')$, then $x = \varphi(y)$ is a stationary point for the problem $\min_{x' \in \Delta^{d-1}} f(x')$.

Show that the lift φ satisfies the $2 \Rightarrow 1$ property. You may find it easier to first treat the case $d = 2$.

Hint: The tangent cone of the standard simplex has the following expression:

$$T_x \Delta^{d-1} = \{v \in \mathbb{R}^d \mid \sum_{i=1}^{d} v_i = 0 \text{ and } v_i \geq 0 \text{ for } i \notin \operatorname{supp}(x)\},$$

where for a point $x \in \Delta^{d-1}$, the support is $\operatorname{supp}(x) = \{i \mid x_i > 0\}$.
(This can be shown by using that for a convex set S, it holds that $\mathrm{T}_x\, S = \overline{\mathrm{K}_x\, S}$, where the cone of feasible directions $\mathrm{K}_x\, S$ is defined as $\mathrm{K}_x\, S = \{\alpha(y - x) \mid y \in S, \alpha \geq 0\}$. See Section 9.1 in these lecture notes for details.)